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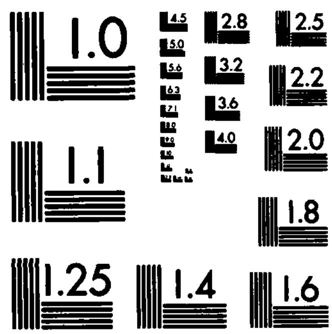

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ON CUMULATIVE SUM PROCEDURES AND A STOPPED WIENER PROCESS FORMULA

by

Rasul A. Khan

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Abstract

Let $X_1, X_2, \ldots, X_{k-1}, X_k, X_{k+1}, \ldots$ be independent random variables such that $X_1, X_2, \ldots, X_{k-1}$ are iid $N(0,\sigma^2)$ and $X_k, X_{k+1}, \ldots$ are iid $N(\mu,\sigma^2)$, $\sigma > 0$, where σ is known and k is an unknown time index of a possible change in distribution. For detecting changes in μ three types of cumulative sum (cusum) procedures are considered. The first one is a class of cusum-type procedures τ such that $E_0\tau = +\infty$ and $E_\mu\tau < \infty$ for $\mu > 0$. The second is a modification of the conventional cusum procedure of Page (1954) which is more efficient. The third is a continuous version T of the modified cusum procedure in terms of Wiener process and its Laplace transform is found which leads to the known results of Taylor (1975) and Nadler and Robbins (1971).

<u>AMS 1970 Subject Classifications</u>: Primary 62L10; Secondary 62L99

<u>Key Words and Phrases</u>: *Detection, cumulative sum (cusum), average run length (ARL), Wiener process, Laplace transform.*

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1. Introduction

Random samples of size m are taken at regular intervals from a production process and sample means $X_1, X_2, \ldots$ are computed. It is assumed that $X_1, X_2, \ldots$ are independent random variables having normal distribution with mean μ and known variance σ^2. The mean μ is said to be in control if $\mu = \mu_0 (\mu \le \mu_0)$ and out of control if $\mu > \mu_0$. There is no loss of generality in assuming that $\mu_0 = 0$. Thus $X_1, X_2, \ldots, X_{k-1}, X_{k+1}, \ldots$ are assumed to be independent random variables such that $X_1, \ldots, X_{k-1}$ are iid $N(0,\sigma^2)$ and $X_k, X_{k+1}, \ldots$ are iid $N(\mu,\sigma^2)$, $\mu > 0$, where σ is known and k is an unknown time index of a possible change in distribution. The oldest method for detecting changes in μ is the Shewart (1931) control chart. Motivated by Wald's sequential probability ratio test and a desire for quick detection Page (1954) defined the following cumulative sum procedure. Let $Y_i = X_i - r$, $r \ge 0$, and set $S_0 = 0$, $S_n = \sum_{i=1}^{n} Y_i$, $W_0 = 0$,

$W_n = \max(0, W_{n-1} + Y_n)$, $n \ge 1$. For $h > 0$ Page's (1954) one-sided cusum procedure is defined by the stopping variable

$$t = \inf\{n\ge 1: W_n \ge h\} = \inf\{n: S_n - \min_{0\le i\le n} S_i \ge h\} \quad , \tag{1.1}$$

and a corrective action is taken at $W_t \ge h$.

The average run length (ARL) is defined to be $E_\mu t$ before the corrective action is taken while the mean has remained at a constant level μ. The rationale for this definition is as follows. Let $P_\mu^{(k)}$ denote the probability under which $X_1, \ldots, X_{k-1}$ are iid $N(0,\sigma^2)$ and $X_k, X_{k+1}, \ldots$ are iid $N(\mu,\sigma^2)$, $\mu > 0$, where σ is known. Thus $P_0 = P_{0,\mu}^{(\infty)}$ entails the model of no change and $P_\mu = P_\mu^{(1)}$ means a change right from the start. Let $E_\mu^{(k)}$ denote the expectation under $P_\mu^{(k)}$. One would like to define a detection stopping variable τ such that $\sup_{k\ge 1} E_\mu^{(k)}((\tau-k+1)|\tau>k-1)$ is minimum subject to $E_\mu^{(\infty)}\tau = E_0\tau \ge A$, where A is a preassigned positive constant. It turns out that t defined by (1.1) has the property

$$\sup_{k\ge 1} E_\mu^{(k)}((t-k+1)|t>k-1) = E_\mu^{(1)} t = E_\mu t \quad . \tag{1.2}$$

To see this we observe that

$$\sup_{k\ge 1} E_\mu^{(k)}((t-k+1)|t>k-1) \ge E_\mu^{(1)} t = E_\mu t \quad . \tag{1.3}$$

Next, note that t can be written as

$$t = \inf\{n\geq 1: \max_{0\leq j<n} \sum_{i=j+1}^{n} Y_i \geq h\} \quad ,$$

and define the cusum procedure t_k in terms of $Y_k, Y_{k+1}, \ldots$ as

$$t_k = \inf\{n\geq k: S_n - \min_{k-1\leq j\leq n} S_j \geq h\} \quad ,$$

where $S_n = \sum_{i=k}^{n} Y_i = S_n(k)$, $n\geq k$.

Then $t = \inf_{k\geq 1} (t_k+k-1) \leq t_k + k - 1$, and hence

$$\sup_{k\geq 1} E_\mu^{(k)} ((t-k+1)|t>k-1) \leq \sup_{k\geq 1} E_\mu^{(k)} t_k = E_\mu^{(1)} t_1 = E_\mu t \quad ,$$

which combined with (1.3) jusitifes (1.2) and the definition of ARL.

There is vast amount of lieterature on the cusum procedure (1.1). The constant r is basically a design constant so as to minimize ARL(μ_1) at a fixed $\mu_1 \geq 0$ subject to $E_0 t \geq A$. This problem has been treated by Ewan and Kemp (1960) in the normal case and by Khan (1978) for the general family of exponential distributions. Unfortunately Page's cusum procedure and the Shewhart control chart as well as the moving average procedure of Lai (1974) have finite ARL when $\mu = 0$. However, in some problems a more desirable property would be $E_0\tau = +\infty$ while $E_\mu\tau < \infty$ for $\mu>0$, which should be as small as possible or at least fares well when compared with the conventional procedures. A trivial S_n procedure with infinite ARL when $\mu = 0$ is given by

$$t_1 = \inf\{n\geq 1: S_n \geq h\} \quad ,$$

where $S_n = \sum_{i=1}^{n} Y_i$, $Y_i = X_i - r$, $r > 0$.

However, $E_\mu t_1 = +\infty$ for $0 < \mu < r$ so that small changes cannot be detected by t_1 . thus it is desireable to develop a cusum-type detection procedure with the above mentioned properties.

A summary of this paper is in order. In Section 2 we develop a class of cusum-type detection procedures τ such that $E_0\tau = +\infty$ and $E_\mu\tau < \infty$ for $\mu > 0$. A modification of the cusum procedure is given in Section 3 and there is numerical evidence that the modified procedure is more efficient that the conventional cusum

procedure. Finally, in Section 4 we discuss a continuous version of the modified cusum procedure in terms of Wiener process and obtain its Laplace transform which leads to simple alternative derivations of some of the results of Taylor (1975) and Nadler and Robbins (1971).

2. A Cusum-Type Procedure τ with $E_0\tau = +\infty$

We will use the likelihood ratio and the mixing techniques of Robbins (1970) to define and study a class of cusum-type procedures τ with the property $E_0\tau = +\infty$ and $E_\mu\tau < \infty$ for $\mu>0$. Let $X_1, X_2, \ldots$ be independent normal random variables with the mean μ and known variance σ^2. The mean μ is in control if $\mu \leq 0$ and $\mu > 0$ indicates lack of control. Clearly, it is enough to consider $\mu = 0$ versus $\mu > 0$ and assume that $\sigma = 1$. Let $P_\mu^{(k)}$ denote the probability under which $X_1, \ldots, X_{k-1}$ are iid $N(0,1)$ and $X_k, X_{k+1}, \ldots$ are iid $N(\mu,1)$ $(\mu>0)$ random variables where k is an unknown time index for a possible change in distribution. Obviously, $P_\mu^{(k)}(A) = P_0(A)$ if $A \in \mathcal{B}(X_1,\ldots,X_{k-1})$ and $P_\mu^{(k)}(A) = P_\mu(A)$ if $A \in \mathcal{B}(X_k,X_{k+1},\ldots)$, where P_μ denotes $N(\mu,1)$ probability measure. If the sequence $X_1,\ldots,X_n$ is observed, its joint probability density function under $P_\mu^{(k)}$ is given by

$$f_{k,n} = f_{0,n} = \prod_{i=1}^{n} \phi(X_i) \qquad , \quad \text{if} \quad n<k$$

$$= \prod_{i=1}^{k-1} \phi(X_i) \prod_{i=k}^{n} \phi(X_i-\mu) \quad , \quad \text{if} \quad n\geq k \quad ,$$

where $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

Any sensible procedure for detecting changes in μ would compare the likelihood that a possible change has occurred at some $k(1\leq k\leq n)$ within the observed segment $(X_1,\ldots X_n)$ versus the possibility that it will occur in the future $(k>n)$. But this means that such a procedure must be based on the ratio

$$Z_{n,k}(\mu) = f_{k,n}\, f_{0,n}^{-1} = [\prod_{i=1}^{k-1} \phi(X_i) \prod_{i=k}^{n} \phi(X_i-\mu)]/\prod_{i=1}^{n} \phi(X_i)$$

$$= \exp(\mu\, S_{n,k} - (1/2)\mu^2\, (n-k+1)) \quad ,$$

where $S_{n,k} = \sum_{i=k}^{n} X_i$.

It is easy to define a cusum procedure based on $Z_{n,k}(\mu)$ if μ is known (c.f. Khan (1979b)). However, since μ is unknown, a possible approach is to use a suitable mixing of $Z_{n,k}(\mu)$, with respect to a distribution function (df) $F(\mu)$ (cf. Robbins (1970)). This is exactly what leads to a class of cusum-type procedures given below.

Clearly, $Z_{n,k}(\mu)$ is a P_0-martingale relative to $\mathcal{F}_{n,k} = \mathcal{B}(X_k,\ldots,X_n)$, $n\geq k$, with $E_0 Z_{n,k}(\mu) = 1$. Now define

$$\zeta_{n,k} = \int_{-\infty}^{\infty} Z_{n,k}(\mu)\,dF(\mu) \quad ,$$

where $F(\mu)$ is a df on $(-\infty,\infty)$.

Then $\zeta_{n,k}$ is also a P_0-martingale relative to $\mathcal{F}_{n,k}$, $n\geq k$. Let b_k be an increasing sequence of positive constants and define

(2.1) $$\tau = \inf\{n\geq 1\colon \zeta_{n,k} \geq b_k \text{ for some } 1\leq k\leq n\} \quad .$$

If $b_k = b>0$, then τ reduces to

$$\tau_1 = \inf\{n\geq 1\colon \max_{1\ k\ n} \zeta_{n,k} \geq b\} \quad ,$$

a procedure studied by Pollak and Siegmund (1975). However, our interest is in τ , which attains $E_0\zeta = +\infty$ by a proper choice of b_k .

Since $\{\zeta_{n,k}\ , \mathcal{F}_{n,k}\ , n\geq k\}$ is a positive martingale with $E_0\zeta_{n,k} = 1$, it follows from martingale inequality that

(2.2) $$P_0(\max_{n\geq k} \zeta_{n,k} \geq b_k^{-1}) \leq b_k^{-1} \quad .$$

Define

$$\tau_k = \inf\{n\geq k\colon \zeta_{n,k} \geq b_k\} \quad .$$

Thus, one obtains from (2.2) that

$$P_0(\tau<\infty) \leq \sum_{k=1}^{\infty} P_0(\tau_k<\infty) = \sum_{k=1}^{\infty} P_0(\max_{n\geq k} \zeta_{n,k} \geq b_k)$$

$$\leq \sum_{k=1}^{\infty} b_k^{-1} \leq \eta < 1 \quad ,$$

by a proper choice of b_k , e.g., with $b_k = \alpha^k$, $\alpha > 1$, $\sum_{k=1}^{\infty} b_k^{-1} = (\alpha-1)^{-1} < 1$.

Since $P_0(\tau<\infty) \leq \eta < 1$, $E_0\tau = +\infty$. Moreover, using an argument in Pollak and Siegmund (1975) it can easily be seen that $E_\mu \tau < \infty$ for $\mu>0$.

We choose $b_k = b_k(\alpha) \to \infty$ as $\alpha \to \infty$ such as above and all the asymptotics are as $\alpha \to \infty$. To obtain an asymptotic upper bound for $E_\mu \tau$ define

$$\tau_k = \inf\{n \geq k: \zeta_{n,k} \geq b_k(\alpha)\} ,$$

and note that $\tau = \inf_{k\geq 1}(\tau_k + k - 1) \leq \tau_1$, so that $E_\mu \tau \leq E_\mu \tau_1$. Assuming the existence of $F'(\mu)$ it follows from a result of Pollak and Siegmund (1975) that

$$\text{(2.3)} \quad E_\mu \tau_1 \underset{\sim}{\approx} [2 \log b_1(\alpha) + \log (\frac{2 \log b_1(\alpha)}{\mu^2}) - \log(2\pi(F'(\mu))^2) - 1]/\mu^2 ,$$

which is really an asymptotic upper bound for $E_\mu \tau$.

<u>Example 1</u>. Let $F'(\mu) = \phi(\mu) = (2\pi)^{-1/2} \exp(-\mu^2/2)$. Then

$$\zeta_{n,k} = (n-k+2)^{-1/2} \exp(S_{n,k}^2/2(n-k+2)) , \quad S_{n,k} = \sum_{i=k}^{n} X_i ,$$

and taking $b_k = \exp(a_k^2/2)$ where $a_k = \sqrt{2k \log\alpha}$, $\alpha>1$, we have

$$\begin{aligned} P_0(S_{n,k} \geq a_{n,k} \text{ for some } n\geq k) &\leq P_0(|S_{n,k}| \geq a_{n,k} \text{ for some } n\geq k) \\ &\leq \exp(-a_k^2/2) = \alpha^{-k} , \end{aligned}$$

where $a_{n,k} = \sqrt{(n-k+2)(a_k^2 + \log(n-k+2))}$.

In fact the first probability is bounded by $(1/2)\exp(-a_k^2/2)$. Moreoever, the one-sided cusum-type procedure τ reduces to

$$\tau = \inf\{n\geq 1: S_{n,k} \geq a_{n,k} \text{ for some } 1\leq k\leq n\} .$$

A two-sided cusum-type procedure can be defined by

$$\tau_0 = \inf\{n\geq 1: |S_{n,k}| \geq a_{n,k} \text{ for some } 1\leq k\leq n\} .$$

It follows from (2.3) that an asymptotic upper bound for $E_\mu \tau$ is

$$\text{(2.4)} \qquad E_\mu \tau \lesssim [2 \log \alpha + \log(\frac{2\log\alpha}{\mu^2}) + \mu^2 - 1]/\mu^2 \text{ as } \alpha\to\infty .$$

Let $\nu_k = \inf\{n\geq k: S_{n,k} \geq a_{n,k}\}$, and since $\tau = \inf_{k\geq 1}(\nu_k+k-1) \leq \nu_1$, $E_\mu\tau \leq E_\mu\nu_1$. Also, since $a_{n,1}$ is an increasing concave boundary, it follows from a result of Robbins (1970) that an upper bound for $E_\mu\nu_1$ is obtained by solving the inequality

$$\text{(2.5)} \qquad \mu E_\mu\nu_1 \leq \sqrt{(2\log\alpha + \log(E_\mu\nu_1))\, E_\mu\nu_1} + \frac{\phi(\mu)}{\Phi(\mu)} + \mu \quad ,$$

where $\Phi(\mu)$ is the standard normal distribution function. The upper bouad for $E_\mu\nu_1$ is in turn an upper bound for $E_\mu\tau$.

<u>Example 2</u>. Let $F'(\mu) = 2\phi(\mu) \quad , \quad \mu>0$

$= 0 \quad , \quad \mu\leq 0$.

In this case $\zeta_{n,k} = 2(n-k+2)^{-1/2}\,\Phi\left(\frac{S_{n,k}}{\sqrt{n-k+2}}\right)\exp(S_{n,k}^2/2(n-k+2))$,

and with $b_k = \alpha^k(\alpha>1)$ (2.1) reduces to

$$\tau = \inf\{n\geq 1: |S_{n,k}|^2 + 2(n-k+2)\log\left(\Phi\left(\frac{S_{n,k}}{\sqrt{n-k+2}}\right)\right) \geq a_{n,k} \text{ for some } 1\leq k\leq n\} \quad ,$$

where $a_{n,k} = (n-k+2)\log(n-k+2) + 2(n-k+2)(k\log\alpha - \log 2)$.
Moreover, it follows from (2.3) that

$$\text{(2.6)} \qquad E_\mu\tau \lesssim [2\log\alpha + \log\left(\frac{2\log\alpha}{\mu^2}\right) + \mu^2 - 1 - 2\log 2]/\mu^2 \text{ as } \alpha\to\infty \quad .$$

The following Tables 1 and 2 are based on (2.4) and (2.6) respectively while Table 3 is based on (2.5).

<u>Table 1</u>

ARL(μ) (Ex.1)

α	μ: $\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\frac{3\sqrt{2}}{2}$	$2\sqrt{2}$
20	71.89	15.95	3.54	2.17	1.59
50	88.69	20.15	5.09	2.64	1.85

Table 2

ARL(μ)(Ex.2)

α	μ				
	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\frac{3\sqrt{2}}{2}$	$2\sqrt{2}$
20	60.80	13.18	3.35	1.87	1.42
50	77.59	17.38	4.40	2.33	1.68

Table 3

Upper Bound for $E_\mu \tau$ (Ex.1)

α	μ				
	$\sqrt{2}/4$	$\sqrt{2}/2$	$\sqrt{2}$	$3\sqrt{2}/2$	$2\sqrt{2}$
20	86	21	5	3	2
50	105	25	6	3	2

Comparing these tables with Table 2 of Lai (1974, p. 138) it is clear that cusum-type procedure obtained by mixture of the likelihood ratio has substantially reduced ARL(μ) in addition to the desirable property of having infinite ARL under no change in distribution.

3. A Modified Cusum Procedure

The cusum-type procedure τ of the last section has the property $E_0\tau = +\infty$ and fairly reduced $ARL(\mu)$ for $\mu>0$. However, $E_\mu\tau = 0(\frac{2\log\alpha}{\mu^2})$ which means that it may take a while to detect small positive changes in μ - perhaps the price for achieving $E_0\tau = +\infty$. In contrast, Page's (1954) cusum procedure t, defined by (1.1) with $r = 0$, has the property that $E_\mu t = 0(h/\mu)$ and $E_0 t \gtrsim h^2$ as $h \to \infty$ (ef. Khan (1979a)). But the comparison favors τ in that $E_0 t < \infty$. Thus it is desirable to somehow improve the cusum procedure (1.1) to increase $E_0 t$ so that the modified version becomes more efficient. Later in Section 4 we will see that this modified version manifests itself into simpler proofs of continuous cusum theory in terms of a Wiener process.

Let $Y_1, Y_2, \ldots$ be independent $N(\mu,\sigma^2)$ random variables with known variance σ^2. As before the target mean is $\mu = 0$ and $\mu > 0$ indicates that the process is out of control. Let $X_i = Y_i - k$, $k \geq 0$, $i = 1,2,\ldots$. Set $W_0 = 0$ and for $b>0$ define

$$W_1 = 0 \quad \text{if } X_1 \leq -b, \qquad W_2 = 0 \quad \text{if } W_1+X_2 \leq -b$$
$$\;\;= X_1 \quad \text{if } X_1 > -b, \qquad\;\; = W_1+X_2 \text{ if } W_1+X_2 > -b,$$
$$W_n = 0 \qquad \text{if } W_{n-1}+X_n \leq -b$$
$$\;\;= W_{n-1}+X_n \qquad \text{if } W_{n-1}+X_n > -b .$$

For any .. > 0 define the cusum procedure

$$N = N(b,h) = \inf\{n \geq 1: W_n \geq h\} ,$$

with a corrective action at $W_N \geq h$. With $b=0$ N becomes the regular cusum procedure. Let $S_n = \sum_{i=1}^{n} X_i$ and define

$$M = \inf\{n \geq 1: S_n \leq -b \text{ or } S_n \geq h\} .$$

A simple renewal argument gives

$$E_\mu N = (E_\mu M)/P_\mu(S_M \geq h) .$$

From Wald's approximations (cf. Khan (1978)) or Wiener process approximation (Section 4) we have

$$(3.1) \qquad E_\mu N \doteq \frac{1}{(\mu-k)} [h - b \frac{(1 - \exp(-2h\gamma))}{(\exp(2b\gamma) - 1)}] \quad , \quad \gamma\neq 0$$

and

$$(3.2) \qquad E_0 N \doteq (h^2 + hb)/\sigma^2 \quad , \quad \gamma=0 \quad ,$$

where $\gamma = (\mu-k)/\sigma^2$.

Letting $b \to 0$ these approximations reduce to the standard approximations to $ARL(\mu)$ for cusum procedure (1.1). Choose $k = 0$ (the purpose of the design constant k has been discussed earlier in Section 1), $\sigma = 1$ and $b = h$. Then (3.1) and (3.2) reduce to

$$(3.3) \qquad E_\mu N(h) \doteq \frac{h}{\mu} - \frac{h}{\mu} \frac{(1 - \exp(-2h\mu))}{(\exp(2h\mu) - 1)} \quad , \quad \mu>0$$

and

$$(3.4) \qquad E_0 N(h) \doteq 2h^2 \quad .$$

Letting N_1 denote Page's (1954) cusum procedure with $k = 0$ and boundary h' it follows that

$$E_\mu N_1(h') \doteq \frac{1}{\mu} [h' - \frac{1}{2\mu} (1 - \exp(-2\mu h'))] \quad , \quad \mu>0$$

and $E_0 N_1(h') \doteq h'^2$.

Setting $h' = h\sqrt{2}$ it follows that

$$(3.5) \qquad E_\mu N_1(h\sqrt{2}) \doteq \frac{1}{\mu} [h\sqrt{2} - \frac{1}{2\mu} (1 - \exp(-2\sqrt{2}h\mu))] \quad , \quad \mu>0$$

and $E_0 N_1(h\sqrt{2}) \doteq E_0 N(h) = 2h^2$.

Thus $N(h)$ and $N_1(h\sqrt{2})$ have about (as $h\to\infty$) the same ARL when $\mu=0$ while their respective $ARL(\mu)$ are given by (3.3) and (3.5). The following tables compare the $ARL(\mu)$ for N and N_1 . These tables show that at least for large h the modified procedure N is more efficient that N_1 .

Table 4

h	ARL(μ)	μ: 0	.01	.02	.1
1	$E_\mu N_1(h\sqrt{2})$	2	1.97	1.96	1.82
	$E_\mu N(h)$	2	1.93	1.94	1.80
3	$E_\mu N_1$	18	17.46	17.02	13.83
	$E_\mu N$	18	17.28	16.86	13.54
5	$E_\mu N_1$	50	47.71	45.59	32.87
	$E_\mu N$	50	47.24	45.31	31.60

Table 5

h	ARL(μ)	μ: 0	.5	1	2
$2h^2=100$	$E_\mu N_1(h\sqrt{2})$	100	18.0	9.5	5.0
	$E_\mu N(h)$	100	14.13	7.07	3.54
$2h^2=590$	$E_\mu N_1$	590	46.58	23.79	12.02
	$E_\mu N$	590	34.35	17.18	8.59
$2h^2=940$	$E_\mu N_1$	940	60.32	30.16	15.21
	$E_\mu N$	940	43.36	21.68	10.84

4. A Stopped Wiener Process Formula

A continuous version of the modified cusum procedure is now consiered and in addition to the approximation formulas used in Section 3 we obtain simple alternative derivations of some of the results of continuous cusum theory of Taylor (1975) and Nadler and Robbins (1971).

Let $\{W(t)\ ,\ W(0)=0\ ,\ t\geq 0\}$ be a Wiener process with a drift parameter μ and scale parameter σ . Let $m(t) = W(t) - \min_{0\leq s\leq t} W(s)$ and $M(t) = \max_{0\leq s\leq t} W(s) - W(t)$.

For $h>0$ define

$$\tau_1 = \inf\{t\geq 0: W(t) - \min_{0\leq s\leq t} W(s) \geq h\} = \inf\{t\geq 0: m(t) \geq h\} \quad , \tag{4.1}$$

$$\tau_2 = \inf\{t\geq 0: \max_{0\leq s\leq t} W(s) - W(t) \geq h\} = \inf\{t\geq 0: M(t) \geq h\} \quad , \tag{4.2}$$

$$\text{and} \quad \tau = \min(\tau_1,\tau_2) = \inf\{t\geq 0: m(t) \geq h \ \text{ or } \ M(t) \geq h\} \quad . \tag{4.3}$$

Here τ_1 and τ_2 are the continuous versions of Page's (1954) one-sided cusum procedures while τ is a continuous version of a symmetric version of two-sided cusum procedure. Taylor (1975) obtained the Laplace transform of τ_2 (hence that of τ_1 also) while Nadler and Robbins (1971) obtained the Laplace transform of τ . Their methods are quite involved due to obvious intrinisc difficulties. However, we consider a continuous version T of the modified cusum procedure and obtain its Laplace transform which lead to simple derivations of the Laplace transforms of τ_1 , τ_2 and τ . In view of the intrinsic difficulties the methods used here show the power of renewal argument and the strnegth of Wald's identity.

The continuous version of the modified cusum procedure is as follows. For $b>0$ and $h>0$ define

$$T_1 = \inf\{t\geq 0: W(t) \leq -b \text{ or } W(t) \geq h\} \quad . \tag{4.4}$$

If T_1 terminates at the lower boundary $-b$, the Wiener process starts from zero all over again and T_1 is repeated. The cycle continues until the upper boundary h is attained. Thus

$$T_2 = \inf\{t\geq 0: W(t+T_1) + b \leq -b \text{ or } W(t+T_1) + b \geq h\} \quad ,$$

... , etc. Clearly, T_1 , T_2 , ... are iid random variables, and the cycles of T_1, T_2 , ... are repeated until the boundary h is hit. By abuse of notation the cycle is terminated by the auxiliary geometric stopping rule

$$N = \inf\{n\geq 1: W(T_n) \geq h\} \quad , \tag{4.5}$$

and a corrective action is taken at T_N . Clearly, the run length is

$$T = T_1 + T_2 + \dots + T_N \quad .$$

Since $T_1, T_2, \dots$ are iid and N has geometric distribution, it follows from Wald's lemma that

$$(4.6) \qquad E_\mu T = E_\mu T_1 E_\mu N = (E_\mu T_1)/P_\mu(W(T_1) \geq h) \quad .$$

First we compute $E_\mu T$ by Wald's identity for Wiener process and then obtain the Laplace transform of T . Let $a(\theta) = \mu\theta + \frac{1}{2}\theta^2\sigma^2$. Then $\{\exp(\theta W(t) - t\, a(\theta))\ ,\ \mathcal{F}_t = \mathcal{B}(W(s)\ ,\ s \leq t)\}$ is a martingale with the property that $E_\mu \exp(\theta W(t) - t\, a(\theta)) = 1$. It is well known that Wald's identity holds for T_1 defined by (4.4). Thus

$$(4.7) \qquad E_\mu \exp(\theta W(T_1) - T_1\, a(\theta)) = 1 \quad .$$

Set $a(\theta) = 0$ giving $\theta = -2\gamma$, where $\gamma = \mu/\sigma^2$, and (4.7) gives

$$E_\mu \exp(-2\gamma\, W(T_1)) = 1 \quad .$$

This identity and the definition of T_1 give

$$(4.8) \qquad P_\mu(W(T_1) \geq h) = p = (\exp(2b\gamma) - 1)/(\exp(2b\gamma) - \exp(-2h\gamma)) \quad ,$$

$$\text{and} \qquad P_\mu(W(T_1) \leq -b) = q = 1-p = (1 - \exp(-2h\gamma))/(\exp(2b\gamma) - \exp(-2h\gamma)) \quad .$$

When $\gamma = 0$ it is easy to see that

$$(4.9) \qquad P_0(W(T_1) \geq h) = p_0 = b/(b+h) \quad ,$$

$$\text{and} \qquad P_0(W(T_1) \leq b) = q_0 = -p_0 = h/(b+h) \quad .$$

Since $E_\mu W(T_1) = \mu\ E\ W(T_1)$ and $E_0 W^2(T_1) = \sigma^2\ E_0\ T_1$, (4.6) , (4.8) and (4.9) give

$$E_\mu T = \frac{h}{\mu} - \frac{b}{\mu}\,\frac{(1-\exp(-2h\gamma))}{(\exp(2b\gamma) - 1)} \quad , \quad \mu \neq 0$$

$$= (h^2+bh)/\sigma^2 \quad , \quad \mu = 0 \ .$$

Letting $b \to 0$ one finds the formula for $E_\mu \tau_1$ (hence $E_\mu \tau_2$ also) which are given by

$$E_\mu \tau_1 = \frac{1}{\mu} [h - \frac{(1-\exp(-2h\gamma))}{2\gamma}] , \quad E_\mu \tau_2 = \frac{1}{\mu} [-h + \frac{(1-\exp(2h\gamma))}{2\gamma}] , \quad \mu \neq 0 ,$$

and

$$E_0 \tau_1 = E_0 \tau_2 = h^2/\sigma^2 .$$

We will now find the Laplace transform of T. To this end, we determine the conditional and unconditional Laplace transforms of T_1. Set $a(\theta) = s (s \geq 0)$ and find the two roots as

$$\theta_+ = -\gamma + \delta \text{ and } \theta_- = -(\gamma + \delta) ,$$

where $\delta = \sqrt{\gamma^2 + (2s/\sigma^2)}$.

Hence it follows from (4.7) that

$$E_\mu \exp(\theta_+ W(T_1) - s\, T_1) = E_\mu \exp(\theta_- W(T_1) - s\, T_1) = 1 . \tag{4.10}$$

Let $g_1 = g_1(s) = E_\mu [\exp(-s\, T_1) | W(T_1) = -b] q$,

and $g_2 = g_2(s) = E_\mu [\exp(-s\, T_1) | W(T_1) = h] p$,

where $p (q = 1-p)$ is defined by (4.8).

Using the definition of T_1 we find from (4.10) that

$$e^{-b\theta^+} g_1 + e^{h\theta^+} g_2 = e^{-b\theta^-} g_1 + e^{h\theta^-} g_2 = 1 ,$$

and after some algebra the solutions are

$$\begin{aligned} g_1 &= e^{-b\gamma} \sinh(h\delta)/\sinh((b+h)\delta) , \\ \text{and } g_2 &= e^{h\gamma} \sinh(b\delta)/\sinh((b+h)\delta) . \end{aligned} \tag{4.11}$$

Hence

$$\phi_0(s) = E_\mu e^{-sT_1} = g_1 + g_2 = (e^{-b\gamma} \sinh(h\delta) + e^{h\gamma} \sinh(b\delta))/\sinh((b+h)\delta) . \tag{4.12}$$

Now recall that $T = T_1 + \ldots + T_N$, where $T_1, T_2, \ldots$ are iid and N has a geometric distribution given by

$$P(N=n) = pq^{n-1} \quad , \; n=1,2,\ldots \; , \; q=1-p \quad ,$$

where p is given by (4.8) <u>if</u> $\mu \neq 0$ and by (4.9) <u>if</u> $\mu = 0$.
From (4.11) we have

$$(4.13) \qquad E_\mu(e^{-sT_1} | W(T_1) = -b) = g_1/q = \frac{e^{-b\gamma} \sinh(h\delta)}{q \sinh((b+h)\delta)}$$

and

$$(4.14) \qquad E_\mu(e^{-sT_1} | W(T_1) = h) = g_2/p = \frac{e^{h\gamma} \sinh(b\delta)}{p \sinh((b+h)\delta)} \quad .$$

Now it follows from the definition of T that

$$(4.15) \qquad E_\mu e^{-sT} = \sum_{n=1}^{\infty} E_\mu(e^{-s(T_1+\ldots T_n)} | N=n) \; P(N=n) \quad .$$

Using conditional independence we have

$$(4.16) \qquad E_\mu(e^{-s(T_1+\ldots T_n)} | N=n) = E_\mu(e^{-sT_1} | N=n) E_\mu(e^{-sT_2} | N=n) \ldots E_\mu(e^{-sT_n} | N=n) \quad .$$

Moreover, it follows from (4.13) and (4.14) and the strong Markov property that

$$(4.17) \qquad E_\mu(e^{-sT_1} | N=n) = E_\mu(e^{-sT_1} | W(T_1) = -b) = \frac{e^{-b\gamma} \sinh(h\delta)}{q \sinh((b+h)\delta)} \quad ,$$

and

$$(4.18) \qquad E_\mu(e^{-sT_n} | N=n) = E_\mu(e^{-sT_n} | W(T_n) = h) = \frac{e^{h\gamma} \sinh(b\delta)}{p \sinh((b+h)\delta)} \quad .$$

It follows from (4.15), (4.16), (4.17) and (4.18) that

$$E_\mu e^{-sT} = \sum_{n=1}^{\infty} \frac{q^{-(n-1)} e^{-(n-1)b\gamma} (\sinh(h\delta))^{n-1}}{(\sinh((b+h)\delta)^{n-1}} \cdot \frac{e^{h\gamma} \sinh(b\delta)}{p \sinh((b+h)\delta)} \cdot p \, q^{n-1}$$

$$= \frac{e^{h\gamma} \sinh(b\delta)}{\sinh((b+h)\delta)} \cdot \frac{1}{1 - \dfrac{e^{-b\gamma} \sinh(h\delta)}{\sinh((b+h)\delta)}} \quad .$$

Hence the Laplace transform of T is given by

$$L_b(s) = E_\mu e^{-sT} = \frac{e^{h\gamma}\sinh(b\delta)}{\sinh((b+h)\delta) - e^{-b\gamma}\sinh(h\delta)} ,$$

and
$$\lim_{b\to 0} L_b(s) = \phi_1(s) = \frac{\delta e^{h\gamma}}{\delta\cosh(h\delta) + \gamma\sinh(h\delta)} , \quad \mu\neq 0$$
$$= 1/\cosh(h\sqrt{2s/\sigma^2}) , \quad \mu=0 ,$$

which is the Laplace transform of τ_1 defined by (4.1). Since τ_2 in (4.2) is representable as τ_1 if $W(t)$ is replaced by $-W(t)$, replacing μ by $-\mu$ in $\phi_1(s)$ we obtain the Laplace transform of τ_2 as

$$\phi_2(s) = \frac{\delta e^{-h\gamma}}{\delta\cosh(h\delta) - \gamma\sinh(h\delta)} , \quad \mu\neq 0$$
$$= 1/\cosh(h\sqrt{2s/\sigma^2}) , \quad \mu=0 ,$$

a result due to Taylor (1975).

We now turn to the problem of finding the Laplace transform of $\tau = \min(\tau_1,\tau_2) = \inf\{t\geq 0: m(t)\geq h \text{ or } M(t)\geq h\}$ defined by (4.3). If $m(\tau)\geq h$ or $M(\tau)\geq h$, then it is easy to verify that $M(\tau)=0$ or $m(\tau)=0$ respectively. Thus $M(\tau_1)=0$ if τ_1 is the first to stop, and $m(\tau_2)=0$ if τ_2 is the first to stop. Using this "starting from scratch" property and using the argument of Khan (1981) we have

<u>Lemma 1</u>. $P(\tau_1>\tau_2) = E\tau_1/(E\tau_1+E\tau_2)$, $P(\tau_1<\tau_2) = E\tau_2/(E\tau_1+E\tau_2)$,

and $E\tau = (E\tau_1 E\tau_2)/(E\tau_1+E\tau_2)$.

Substituting the expressions for $E\tau_1$ and $E\tau_2$ (given earlier) in Lemma 1 one obtains the formula for $E\tau$ found by Nadler and Robbins (1971).

Let $\phi(s) = Ee^{-s\tau}, \phi(0)=1$, be the Laplace transform of τ. Using the "starting from scratch" property and repeating the discrete argument of Khan (1981) in the continuous case we have the identity

$$\phi(s) = \frac{\phi_1(s) + \phi_2(s) - 2\phi_1(s)\phi_2(s)}{1 - \phi_1(s)\phi_2(s)} , \quad s > 0 \quad . \tag{4.9}$$

A substitution of the expressions for $\phi_1(s)$ and $\phi_2(s)$ in (4.19) and some calculations give

$$\phi(s) = \frac{\delta}{(\delta^2-\gamma^2)\ \sinh^2(h\delta)} [(\delta-\gamma)\cosh(h(\delta+\gamma)) + (\delta+\gamma)\cosh(h(\delta-\gamma))-2\delta] \ , \ \mu \neq 0$$

$$= \operatorname{sech}^2(h\sqrt{2s/\sigma^2} \quad , \quad \mu = 0 \quad ,$$

a result due to Nadler and Robbins (1971).

Thus the continuous version of the modified cusum procedure gives the approximations to $\mathrm{ARL}(\mu)$ used in Section 3 and provides simple derivations of the Laplace transforms of τ_1, τ_2 and τ which were obtained by Taylor (1975) and Nadler and Robbins (1971) by difficult methods.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $X_1, X_2, \ldots, X_{k-1}, X_k, X_{k+1}, \ldots$ be independent random variables such that $X_1, X_2, \ldots, X_{k-1}$ are iid $N(0,\sigma^2)$ and $X_k, X_{k+1}, \ldots$ are iid $N(\mu,\sigma^2)$, $\sigma > 0$, where σ is known and k is an unknown time index of possible change in distribution. For detecting changes in μ three types of cumulative sum (cusum) procedures are considered. The first one is a class of cusum-type procedures τ such that $E_0\tau = +\infty$ and $E_\mu\tau < \infty$ for $\mu>0$.

20. Abstract (cont'd)

The second is a modification of the conventional cusum procedure of Page (1954) which is more efficient. The third is a continuous version T of the modified cusum procedure in terms of Wiener process and its Laplace transform is found which leads to the known results of Taylor (1975) and Nadler and Robbins (1971).

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